Dirichlet Series of Squares of Sums of Squares

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**Abstract.** Hardy & Wright records elegant forms for the generating functions of the divisor functions \( \sigma_k(n) = \sum_{d|n} d^k \) and \( \sigma^2_k(n) \):

\[
\begin{align*}
(1) & \quad \sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k) \\
(2) & \quad \sum_{n=1}^{\infty} \frac{\sigma^2_k(n)}{n^s} = \frac{\zeta(s)\zeta(s-k)^2\zeta(s-2k)}{\zeta(2s-2k)}.
\end{align*}
\]

- We have extended this elegant pair to:

**Theorem 1** For completely multiplicative \( f_1, f_2 \) and \( g_1, g_2 \),

\[
(3) \quad \sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n) \cdot (f_2 * g_2)(n)}{n^s} = \frac{L_{f_1f_2}(s)L_{g_1g_2}(s)L_{f_1g_2}(s)L_{g_1f_2}(s)}{L_{f_1f_2g_1g_2}(2s)}
\]

where \( f * g(n) := \sum_{d|n} f(d)g(n/d) \) and \( L_f(s) := \sum_{n=1}^{\infty} f(n)n^{-s} \) is a Dirichlet series.
• Let \( r_N(n) \) be the number of solutions of \( x_1^2 + \cdots + x_N^2 = n \) and let \( r_{2, P}(n) \) be the number of solutions of \( x^2 + Py^2 = n \).

◊ One application of Theorem 1 is to obtain closed forms, in terms of \( \zeta(s) \) and Dirichlet \( L \)-functions, for the generating functions of functions such as \( r_N(n), r_N^2(n), r_{2, P}(n) \) and \( r_{2, P}(n)^2 \) for certain \( P \) and (even) \( N = 2, 4, 6, 8 \).

• We also use these generating functions to obtain asymptotics for the average values of each function for which we obtain a Dirichlet series.

• We finish by discussing the more vexing case \( N = 3 \), and related matters.

This is joint work with Stephen Choi, SFU, (Rankin Memorial Volume, Ramanujan Journal, in press) who will describe the asymptotics in more detail.
OUTLINE

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1. MOTIVATION and BACKGROUND

Evaluating residues of the corresponding g.f.s at their largest real poles and Cauchy’s integral theorem leads to

$$\sum_{n \leq x} r_N^2(n) = W_N x^{N-1} + O(x^{N-5/3+\epsilon})$$

for $N = 4, 6, 8$ where for $N \geq 3$

(4)

$$W_N := \frac{1}{(N - 1)(1 - 2^{-N})} \frac{\pi^N}{\Gamma^2(N/2)} \frac{\zeta(N - 1)}{\zeta(N)}$$

• This establishes some initial cases of a conjecture due to Wagon: for $N \geq 3$

Conjecture (W)  

$$\sum_{n < x} r_N^2(n) \sim W_N x^{N-1}.$$  

• This conjecture triggered our interest in such series representations.
Recently, Crandall and Wagon proved that

$$\lim_{x \to \infty} x^{1-N} \sum_{n<x} r_N^2(n) = W_N,$$

with various rates of convergence ($N = 3$ is especially difficult and slow). In their treatment they needed to evaluate

$$\sum_{n=1}^{\infty} \frac{\phi(n)\sigma_0(n^2)}{n^s}$$

and we have established, by an easier version of what follows, that it is

$$\sum_{n=1}^{\infty} \frac{\phi(n)\sigma_0(n^2)}{n^s} = \zeta^3(s-1) \prod_p \left(1 - \frac{3}{p^s} - \frac{1}{p^{2s-2}} + \frac{4}{p^{2s-1}} - \frac{1}{p^{3s-2}} \right)$$

where the product is over all primes.
• Why study the second-order summatories? Partly, since (W) implies that sums of three squares have positive density. Though the density of \( S = \{x^2 + y^2 + z^2\} \) is known to be exactly 5/6, there are intriguing signal-processing and analytic notions that lead more easily at least to positivity of said density.

• Indeed, from Cauchy-Schwarz, we have

\[
\#\{n < x; n \in S\} > \frac{(\sum_{n<x} r_3(n))^2}{\sum_{n<x} r_3^2(n)},
\]

so (W) gives an explicit lower bound on the density of \( S \). Of course, the density for sums of more than 3 squares is likewise positive, and boundable — but Lagrange’s theorem dominates.

◊ Still, the signal-processing and computational notions of Crandall and Wagon forge an attractive link between these \( L \)-series of our current interest and additive number theory.
The 'Entropy Inequality' below might give improved bounds. For any non-negative integer sequence \( a_n \):

\[
\# \{ 1 \leq n \leq x : a_n > 0 \} \geq \frac{\sum_{n \leq x} a_n}{\prod_{n \leq x} a_n / \sum_{n \leq x} a_n} \geq \left(\frac{\sum_{n \leq x} a_n}{\sum_{n \leq x} a_n^2}\right)^2.
\]

Let \( A \) be the arithmetic mean of \( n \)-terms and \( A_p \) be the \( p \)-th Hölder mean. Let

\[
R_p := (\frac{A}{A_p})^q,
\]

with \( q := p/(p-1) \). Now letting \( p \to 1 \) produces

\[
R_1 := \frac{A}{G^*}
\]

where

\[
G^*(a_1, a_2, \cdots, a_n) := a_1^{a_1/A} a_2^{a_2/A} \cdots a_n^{a_n/A}.
\]

By the A-G inequality, \( R_1 \geq R_2 \) which is (5).

This entails study of

\[
\sum_{n > 0} \frac{a_n \log a_n}{n^s}.
\]
The proof of Theorem 1 relies on direct manipulation and happy simplifications of the underlying *Euler products*, as illustrated in §2.

1. A first easy application of Theorem 1 is to
\[
\sum_{n=1}^{\infty} \sigma_k(n)n^{-s} \quad \text{and} \quad \sum_{n=1}^{\infty} \sigma_a(n)\sigma_b(n)n^{-s}.
\]

We let \( f_1(n) := n^k \), \( f_2(n) := \delta(n) \) and \( g_1(n) = g_2(n) := 1 \) (\( \delta(n) \) is 1 if \( n = 1 \) and 0 otherwise). Then
\[
L_{f_1 f_2}(s) = L_{g_1 f_2}(s) = L_{f_1 f_2 g_1 g_2}(s) = 1,
\]
\[
L_{f_1 g_2}(s) = \zeta(s - k), \quad L_{g_1 g_2}(s) = \zeta(s).
\]

Thus Theorem 1 gives
\[
\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s} = \zeta(s)\zeta(s - k)
\]

which recovers (1).
2. Similarly, if we let \( f_1(n) := n^a, f_2(n) := n^b \) and \( g_1(n) = g_2(n) := 1 \), then

\[
L_{f_1f_2}(s) = L_{f_1f_2g_1g_2}(s) = \zeta(s - (a + b)),
\]

\[
L_{g_1g_2}(s) = \zeta(s),
\]

\[
L_{f_1g_2}(s) = \zeta(s - a), \quad L_{f_2g_1}(s) = \zeta(s - b).
\]

and Theorem 1 gives

\[
\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s - (a + b))\zeta(s)\zeta(s - a)\zeta(s - b)}{\zeta(2s - (a + b))}.
\]

In particular, for any real \( \lambda \),

\[
(6) \quad \sum_{n=1}^{\infty} \sigma_\lambda^2(n)n^{-s} = \frac{\zeta(s - 2\lambda)\zeta(s - \lambda)^2\zeta(s)}{\zeta(2(s - \lambda))}.
\]
2. L-SERIES of 2 SQUARES

Theorem 1 has no ‘closed-form’ extension to higher order. We consider the generating functions for the \(k\)th moment of \(r_2(n)\) (see Connors and Keating, 1999). For \(n, k \geq 0\), let

\[
A(n, k) := \sum_{j=0}^{k} (-1)^j (k - j)^n \binom{n + 1}{j}
\]

and

\[
E_n(x) := \sum_{k=1}^{n} A(n, k) x^{k-1}
\]

where \(E_n(x)\) is the \(n\)th Euler polynomial. The first few are \(E_1(x) = 1\), \(E_2(x) = 1 + x\), \(E_3(x) = 1 + 4x + x^2\) and \(E_4(x) = 1 + 11x + 11x^2 + x^3\). Then \(A(n, k)\) satisfies: \(A(1, 1) = 1\) and

\[A(n, k) = k A(n - 1, k) + (n - k + 1) A(n - 1, k - 1)\]

and so

\[
\sum_{l=0}^{\infty} l^n x^l = \frac{x E_n(x)}{(1 - x)^{n+1}}, \quad n = 1, 2, \cdots.
\]
Equation (7) yields the generating functions for the higher moments of \( r_2(n) \) as follows: for \( \mu \equiv 0 \) or \( 1 \) (mod 4), we let \( \left( \frac{\mu}{n} \right) \) be the *Jacobi symbol* and consider the \( L \)-function

\[
L_{\mu}(s) := \sum_{n=1}^{\infty} \left( \frac{\mu}{n} \right) n^{-s}.
\]

Now Lorenz showed

\[
\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} = 4 \zeta(s)L_{-4}(s) = \sum_{n=1}^{\infty} \frac{4 \left( 1 \ast \left( \frac{-4}{n} \right) \right)(n)}{n^s}
\]

and so \( r_2(n) = 4 \left( 1 \ast \left( \frac{-4}{n} \right) \right)(n) \) for any \( n \geq 1 \). A simple calculation shows that for any \( l \geq 0 \),

\[
\left( 1 \ast \left( \frac{-4}{n} \right) \right)(p^l) = \begin{cases} 
1 & \text{if } p = 2; \\
 l + 1 & \text{if } p \geq 3, \left( \frac{-1}{p} \right) = 1; \\
\frac{(-1)^l+1}{2} & \text{else}.
\end{cases}
\]
We now have

\[
\sum_{n=1}^{\infty} \frac{r_2^N(n)}{n^s} = 4^N \sum_{n=1}^{\infty} \frac{(1 * \left(\frac{-4}{n}\right))(n)}{n^s}
\]

\[
= 4^N \prod_p \sum_{l=0}^{\infty} \frac{\left(1 * \left(\frac{-4}{n}\right)\right)\left(p^l\right)}{p^{ls}}
\]

\[
= \frac{4^N}{1 - 2^{-s}} \left\{ \prod \sum_{l=0}^{\infty} \left(\frac{-1}{p}\right)^l + 1 \right\}^N p^{-ls}
\]

\[
\times \left\{ \prod \sum_{l=0}^{\infty} (l + 1)\left(p^{-ls}\right) \right\}
\]

\[
= \frac{4^N}{1 - 2^{-s}} \prod \frac{1}{1 - p^{-2s}} \prod \frac{E_N(p^{-s})}{(1 - p^{-s})^{N+1}}
\]

on using (7).
• When $N = 2$, we have most pleasingly,

$$\sum_{n=1}^{\infty} \frac{r_2(n)}{n^s} =$$

$$\frac{16}{1 - 2^{-s}} \prod \left( \frac{-1}{p} \right) = -1 \prod \left( \frac{-1}{p} \right) = 1 \prod \frac{1 + p^{-s}}{(1 - p^{-s})^3}$$

$$= \frac{(4\zeta(s)L_{-4}(s))^2}{(1 + 2^{-s})\zeta(2s)}. \quad (8)$$

• The other generating functions don’t evaluate as $L$-functions as completely. Thus,

$$\sum_{n=1}^{\infty} \frac{r_3(n)}{n^s} =$$

$$\frac{64}{1 - 2^{-s}} \prod \left( \frac{-1}{p} \right) = -1 \prod \left( \frac{-1}{p} \right) = 1 \prod \frac{1 + 4p^{-s} + p^{-2s}}{(1 - p^{-s})^4},$$

$$\sum_{n=1}^{\infty} \frac{r_4(n)}{n^s} = \frac{256}{1 - 2^{-s}}$$

$$\prod \left( \frac{-1}{p} \right) = -1 \prod \left( \frac{-1}{p} \right) = 1 \prod \frac{1 + 11p^{-s} + 11p^{-2s} + p^{-3s}}{(1 - p^{-s})^5}$$
3. L-SERIES of 4, 6, 8 SQUARES

• Whenever a Dirichlet series is expressible as a sum of two-fold products of \( L \)-functions:

\[
L_f(s) = \sum_{\chi_1, \chi_2} a(\chi_1, \chi_2)L_{\chi_1}(s)L_{\chi_2}(s),
\]

we are able to provide a closed form (in terms of \( L \)-functions) of the Dirichlet series \( L_{f_2}(s) = \sum_{n=1}^{\infty} f^2(n)n^{-s} \), on using Theorem 1.

• In particular, let \( r_N(n) \) be the number of solutions to \( x_1^2 + x_2^2 + \cdots + x_N^2 = n \) (counting permutations and signs) and let

\[
L_N(s) := \sum_{n=1}^{\infty} r_N(n)n^{-s}, \quad R_N(s) := \sum_{n=1}^{\infty} r_N^2(n)n^{-s}
\]

be the Dirichlet series corresponding to \( r_N(n) \) and \( r_N^2(n) \).

◊ Closed forms are obtainable, via the Mellin transform, for \( L_N(s) \) for certain even \( N \) — from the explicit formulae known for \( r_N(n) \).
For example, we have

\[ L_2(s) = 4\zeta(s)\beta(s), \]

\[ L_4(s) = 8(1 - 4^{1-s})\zeta(s)\zeta(s - 1), \]

\[ L_6(s) = 16\zeta(s - 2)\beta(s) - 4\zeta(s)\beta(s - 2), \]

\[ L_8(s) = 16(1 - 2^{1-s} + 4^{2-s})\zeta(s)\zeta(s - 3), \]

where \( \beta(s) := L_{-4}(s) = \sum_{n \geq 0}(-1)^n(2n + 1)^{-s}. \)

Theorem 1 lets us obtain counterpart closed forms for \( R_N(s) \) from the above expressions for \( L_N(s) \).

\( R_2(s) \). We saw in §2 that

\[ R_2(s) = \sum_{n=1}^{\infty} \frac{r_2^2(n)}{n^s} = \frac{(4\zeta(s)L_{-4}(s))^2}{(1 + 2^{-s})\zeta(2s)} \]

as is directly in Theorem 1 and (9) for \( f_1(n) := f_2(n) := 1 \) and \( g_1(n) := g_2(n) := \left( \frac{-4}{n} \right). \)
• $R_6(s)$. Write

$$L_6(s) = 16\zeta(s-2)\beta(s) - 4\zeta(s)\beta(s-2)$$

$$= \sum_{n=1}^{\infty} (16(f_1 * g_1)(n) - 4(f_2 * g_2)(n)) n^{-s}$$

where $f_1(n) := n^2$, $g_1(n) := \left(\frac{4}{n}\right)$, $f_2(n) := 1$ and $g_2(n) := \left(\frac{4}{n}\right) n^2$. It follows from Theorem 1 that

$$R_6(n) = \sum_{n=1}^{\infty} (16(f_1 * g_1)(n) - 4(f_2 * g_2)(n))^2 n^{-s}$$

$$= 16^2 \sum_{n=1}^{\infty} (f_1 * g_1)^2(n) n^{-s}$$

$$- 128 \sum_{n=1}^{\infty} (f_1 * g_1)(n)(f_2 * g_2)(n) n^{-s}$$

$$+ 16 \sum_{n=1}^{\infty} (f_2 * g_2)^2(n) n^{-s}.$$ 

Thus,
\[ R_6(n) = 16^2 \frac{L_{f_1}^2(s)L_{g_1}^2(s)L_{f_1g_1}(s)^2}{L_{f_1g_1}^2(2s)} - 128 \frac{L_{f_1}f_2(s)L_{g_1g_2}(s)L_{f_1g_2}(s)L_{g_1f_2}(s)}{L_{f_1f_2g_2}(2s)} + 16 \frac{L_{f_2}^2(s)L_{g_2}^2(s)L_{f_2g_2}(s)^2}{L_{f_2g_2}^2(2s)}. \]

- It remains to sum the component \( L \)-functions:

\[
L_{f_1}^2(s) = \zeta(s - 4), \quad L_{g_1}^2(s) = (1 - 2^{-s})\zeta(s),
\]

\[
L_{f_2}^2(s) = \zeta(s), \quad L_{g_2}^2(s) = (1 - 16 \cdot 2^{-s})\zeta(s - 4),
\]

\[
L_{f_1g_1}(s) = \beta(s - 2), \quad L_{f_1f_2}(s) = \zeta(s - 2),
\]

\[
L_{g_1g_2}(s) = (1 - 4 \cdot 2^{-s})\zeta(s - 2),
\]

\[
L_{f_1g_2}(s) = \beta(s - 4), \quad L_{g_1f_2}(s) = \beta(s),
\]

\[
L_{f_2g_2}(s) = \beta(s - 2),
\]

\[
L_{f_1g_1}^2(s) = L_{f_2g_2}^2(s) = L_{f_1f_2g_2}(s) = (1 - 16 \cdot 2^{-s})\zeta(s - 4).
\]
Hence

\[ R_6(s) = 16 \frac{(17 - 32 \cdot 2^{-s}) \zeta(s - 4) \beta^2(s - 2) \zeta(s)}{(1 - 16 \cdot 2^{-2s}) \zeta(2s - 4)} - \frac{128 \beta(s - 4) \zeta^2(s - 2) \beta(s)}{(1 + 4 \cdot 2^{-s}) \zeta(2s - 4)}. \]

• For \( R_4(s) \) and \( R_8(s) \), we need the following companion lemma:

**Lemma.** Suppose \( f(n) \) is multiplicative. Let \( p \) be a prime and suppose

\[
\sum_{n=1}^{\infty} \frac{A(n)}{n^s} := \sum_{m=0}^{\infty} \frac{a_m}{p^{ms}} \sum_{n=1}^{\infty} \frac{f(n)}{n^s}
\]

is the product of \( L_f(s) \) and a power series in \( p^{-s} \).
Then

\[
\sum_{n=1}^{\infty} \frac{A^2(n)}{n^s} =
\]

\[
L_{f^2}(s) \sum_{m=0}^{\infty} \frac{a_m^2}{p^{ms}} + 2L_{f^2}(s) \left( \sum_{l=0}^{\infty} \frac{f^2(p^l)}{p^{ls}} \right)^{-1}
\]

\[
x \sum_{k=1}^{\infty} \left\{ \sum_{m=0}^{\infty} \frac{a_{m+k}a_m}{p^{ms}} \right\} \left\{ \sum_{l=0}^{\infty} \frac{f(p^l)f(p^{l+k})}{p^{ls}} \right\} p^{-ks}.
\]

- Applying the Lemma to (9), and using Theorem 1, we have completed the proof of the following Theorem.
Theorem 2 We may write

\[ R_2(s) = \frac{(4\zeta(s)\beta(s))^2}{(1 + 2^{-s})\zeta(2s)}, \quad \Re(s) > 1; \]

\[ R_4(s) = 64(8 \cdot 2^{3-3s} - 10 \cdot 2^{2-2s} + 2^{1-s} + 1) \times \]
\[ \frac{\zeta(s - 2)\zeta^2(s - 1)\zeta(s)}{(1 + 2^{1-s})\zeta(2s - 2)}, \quad \Re(s) > 3; \]

\[ R_6(s) = 16 \frac{(17 - 32 \cdot 2^{-s})\zeta(s - 4)\beta^2(s - 2)\zeta(s)}{(1 - 16 \cdot 2^{-2s})\zeta(2s - 4)} \]
\[ - \frac{128 \beta(s - 4)\zeta^2(s - 2)\beta(s)}{(1 + 4 \cdot 2^{-s})\zeta(2s - 4)}, \quad \Re(s) > 5; \]

and

\[ R_8(s) = 256(32 \cdot 2^{6-2s} - 3 \cdot 2^{3-s} + 1) \times \]
\[ \frac{\zeta(s - 6)\zeta^2(s - 3)\zeta(s)}{(1 + 2^{3-s})\zeta(2s - 6)}, \quad \Re(s) > 7. \]
• Heuristics using (2) suggested the formula for \( R_4 \) to Crandall.

• Rough analysis suggested the formula for \( R_2 \) to me. 'Integer relation methods' found the exact formula.

• Pattern matching and interpolation found the formula for \( R_8 \).

• This also found a few of the identities in §4.

• \( R_6 \) was beyond my reach computationally!
Asymptotics Since $\epsilon \zeta(1 + \epsilon) \to 1$ as $\epsilon \to 0$, 
$\lim_{\epsilon \to 0} \epsilon R_N(N - 1 + \epsilon)$ at its largest pole is:

$$\lim_{\epsilon \to 0} \epsilon R_4(3 + \epsilon) = 96\zeta(3) = 3W_4$$

$$\lim_{\epsilon \to 0} \epsilon R_6(5 + \epsilon) = 240\zeta(5) = 5W_6$$

and

$$\lim_{\epsilon \to 0} \epsilon R_8(7 + \epsilon) = \frac{4480}{17}\zeta(7) = 7W_8.$$ 

- Using the ‘hyperbola method’ and a direct convolution argument, Theorem 2 yields:

**Corollary 3** We have

$$\sum_{n \leq x} r_2^2(n) = 4x \log x + 4\alpha x + O(x^{\frac{2}{3}})$$

where

$$\alpha : = 2\gamma + \frac{8}{\pi} L'_4(1) - \frac{12}{\pi^2} \zeta'(2) + \frac{1}{3} \log 2 - 1 \approx 2.0166216 \ldots$$
and

\[ \sum_{n \leq x} r_N^2(n) = W_N x^{N-1} + O(x^{N-2}) \]

with \( W_N \) given by (4), for \( N = 6, 8 \) and for \( N = 4 \) with error term \( O(x^2 \log^5 x) \).

- For \( N = 4, 6, 8 \), the best possible estimate would appear to be \( O(x^{N-2}) \).

- More generally, for \( N \geq 5 \), from Hardy’s singular series formula for \( r_N(n) \), we prove

**Theorem 4** For \( N \geq 5, N \neq 6 \) and \( x \geq 1 \), we have

\[ \sum_{n \leq x} r_N^2(n) = W_N x^{N-1} + O(x^{N-2} + x^{3N/4}). \]

- Crandall and Wagon have established asymptotics for \( N \geq 3 \).
4. APPLICATIONS to $r_{2,P}$

- There is a rich parallel theory of L-functions over imaginary quadratic fields. In this vein, let $r_{2,P}(n)$ be the number of solutions to $x^2 + Py^2 = n$ (counting sign and order). Denote

$$L_{2,P}(s) := \sum_{n=1}^{\infty} r_{2,P}(n)n^{-s},$$

$$R_{2,P}(s) := \sum_{n=1}^{\infty} r_{2,P}(n)^2n^{-s}.$$ 

- If the quadratic form $x^2 + Py^2$ has disjoint discriminants (has one form per genus), then

$$L_{2,P} = 2^{1-t} \sum_{\mu \mid P} L_{\epsilon_{\mu}\mu}(s)L_{-4P\epsilon_{\mu}/\mu}(s)$$

$$(10) = \sum_{n=1}^{\infty} \left\{ 2^{1-t} \sum_{\mu \mid P} \left( \frac{\epsilon_{\mu}\mu}{n} \right) * \left( \frac{-4P\epsilon_{\mu}/\mu}{n} \right) \right\} n^{-s}$$

where $P$ is an odd square-free number, $t$ is the number of distinct factors of $P$ and $\epsilon_{\mu} := \left( \frac{-1}{\mu} \right)$ (Glasser-Zucker-Robertson).
 Explicitly, (10) holds for all type one numbers. These include and may comprise:

\[ P = 5, 13, 21, 33, 37, 57, 85, 93, 105, 133, 165, 177, 253, 273, 345, 357, 385, 1365. \]

We call such \( P \) solvable.

Using (10), we have

\[
R_{2,P}(s) = \sum_{n=1}^{\infty} 2^{2-2t} \sum_{\mu_1, \mu_2 | P} \left[ \left( \frac{e_{\mu_1, \mu_2}}{n} \right) \ast \left( \frac{-4P e_{\mu_1} / \mu_1}{n} \right) \right] \\
\times \left[ \left( \frac{e_{\mu_2, \mu_2}}{n} \right) \ast \left( \frac{-4P e_{\mu_2} / \mu_2}{n} \right) \right] n^{-s} \\
= 2^{2-2t} \sum_{\mu_1, \mu_2 | P} \sum_{n=1}^{\infty} \left[ \left( \frac{e_{\mu_1, \mu_2}}{n} \right) \ast \left( \frac{-4P e_{\mu_1} / \mu_1}{n} \right) \right] \\
\times \left[ \left( \frac{e_{\mu_2, \mu_2}}{n} \right) \ast \left( \frac{-4P e_{\mu_2} / \mu_2}{n} \right) \right] n^{-s}.
\]

\[ \diamond \text{ Note that } R_{2,P}(s) \text{ is a sum of Dirichlet series in the form of Theorem 1.} \]
• We have similar closed forms of $L$-functions for the quadratic form $x^2 + 2Py^2$ with discriminant $-8P$:

\[ L_{2,2P} = 2^{1-t} \sum_{\mu|P} L_{\epsilon_\mu}(s)L_{-8P\epsilon_\mu}(s). \]

For the type two numbers

\[ P = 1, 3, 5, 11, 15, 21, 29, 35, 39, 51, 65, 95, 105, 165, 231. \]

◊ We note that $210 = 2 \times 105$ yields the elliptic integral invariant, $k_{210}$, which Ramanujan sent to Hardy in his famous letter.

• We can thus evaluate $R_{2,P}$ and $R_{2,2P}$ and obtain the asymptotics analogous to those for $r_2^2(n)$. The prime cases provide:
Corollary 5  We have

\[ R_{2,p}(s) = \frac{2\zeta^2(s)L_{-4p}(s)}{(1 + 2^{-s})(1 + p^{-s})\zeta(2s)} + \frac{2L_{-p}^2(s)L_{-4}(s)}{(1 - 2^{-s})(1 + p^{-s})\zeta(2s)} \]

for \( p = 5, 13, 37 \), while

\[ R_{2,2}(s) = \frac{4\zeta^2(s)L_{-8}(s)}{(1 + 2^{-s})\zeta(2s)}. \]

Similarly,

\[ R_{2,2p}(s) = \frac{2\zeta^2(s)L_{-8p}(s)}{(1 + 2^{-s})(1 + p^{-s})\zeta(2s)} + \frac{2L_{-p}^2(s)L_{-8}(s)}{(1 - 2^{-s})(1 - p^{-s})\zeta(2s)} \]

for \( p = 3, 11 \) while

\[ R_{2,2p}(s) = \frac{2\zeta^2(s)L_{-8p}(s)}{(1 + 2^{-s})(1 + p^{-s})\zeta(2s)} + \frac{2L_{-p}^2(s)L_{-8}(s)}{(1 - 2^{-s})(1 - p^{-s})\zeta(2s)} \]

for \( p = 5, 29 \).
• Closed forms for $L_{2,P}(s)$ are also accessible for some $P$ other than those of type one or type two. Thus,

$$L_{2,3}(s) = (2 + 4^{1-s}) \zeta(s)L_{-3}(s),$$

and Theorem 1 and the Lemma yield

$$R_{2,3}(s) = 4\frac{1 + 2^{3-2s}(\zeta(s)L_{-3}(s))^2}{1 + 3^{-s}} \zeta(2s).$$

• There are some simple closed forms for more general binary quadratic forms. Let

$$L_{(a,b,c)}(s) : = \sum_{n=1}^{\infty} \frac{r_{(a,b,c)}(n)}{n^s}$$

and $R_{(a,b,c)}(s) := \sum_{n=1}^{\infty} \frac{r_{(a,b,c)}(n)^2}{n^s}$ where $r_{(a,b,c)}(n)$ is the number of representations of $n$ by the quadratic form $ax^2 + bxy + cy^2$. 

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• Then, we have (Shanks, 75)

\[ \sum_{h(D)} L_{(a,b,c)}(s) = \omega(D)\zeta(s)L_D(s) \]

summed over the \( h(D) \) inequivalent reduced quadratic forms of discriminant \( D := b^2 - 4ac \) and \( \omega(-3) = 6, \omega(-4) = 4 \) and \( \omega(D) = 2 \) for \( D < -4 \).

• In particular, for \( c = 2, 3, 5, 11, 17, 41 \), the class number \( h(D) = 1 \) and the result is especially simple:

\[ L_{(1,1,c)}(s) = 2\zeta(s)L_D(s). \]

Hence from Theorem 1, we have

\[ R_{(1,1,c)}(s) = \frac{4(\zeta(s)L_D(s))^2}{(1 + |D|^{-s})\zeta(2s)}, \]

with similar formulae for \( (a, b, c) = (1, 1, 1) \) and \( (1, 0, 1) \).
• Thanks to the On-Line Encyclopedia of Integer Sequences

  www.research.att.com/~njas/sequences/

we discover that the sequence 2, 3, 5, 11, 17, 41 is exactly the so-called Euler “lucky number” sequence which consists of the numbers $n$ such that

$$m \rightarrow m^2 - m + n$$

has prime values for $m = 0, \cdots, n - 1$. 
5. THREE, 12 and 24 SQUARES

- Odd squares are notoriously less amenable to closed forms. In this subsection, we primarily record some results for $r_3(n)$, the number of representations of $n$ as a sum of three squares. Following Hardy and Bateman, Hua gives the following formula for $r_3(n)$. Let

$$
\chi_2(n) := \begin{cases} 
0 & \text{if } 4^{-a}n \equiv 7 \pmod{8}; \\
2^{-a} & \text{if } 4^{-a}n \equiv 3 \pmod{8}; \\
3 \cdot 2^{-1-a} & \text{if } 4^{-a}n \equiv 1, 2, 5, 6 \pmod{8}
\end{cases}
$$

where $a$ is the highest power of 4 dividing $n$.

Then

$$
(11) \quad r_3(n) = \frac{16\sqrt{n}}{\pi} L_{-4n}(1) \chi_2(n) \\
\times \prod_{p^2|n} \left( \frac{p^{-\tau} - 1}{p-1} + p^{-\tau} \left(1 - \frac{1}{p} \left(\frac{-p^{-2\tau n}}{p}\right)\right)^{-1}\right)
$$

where $\tau = \tau_p$ is the highest power of $p^2$ dividing $n$. 

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The Dirichlet series for $r_3(n)$ deriving from (11) is not as malleable as those of (9), but we are able to derive a nice expression in terms of Bessel functions.

Let $K_s$ be the modified Bessel function of the second kind. Then we have

$$K_s(x) = \frac{1}{2} \left( \frac{x}{2} \right)^s \int_0^\infty e^{-t - \frac{x^2}{4t}} \frac{dt}{t^{s+1}}.$$  

By the substitution $t = \frac{1}{u}$ in (12), we get

$$K_s(x) = \frac{1}{2} \left( \frac{x}{2} \right)^s \int_0^\infty e^{-\frac{x^2}{4u} - \frac{1}{u}u^{s-1}} du.$$  

Let

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$$

be the classical Jacobean theta function.

In view of the Poisson summation formula, we have, for $t > 0$

$$\theta_3(e^{-\pi t}) = t^{-\frac{1}{2}} \theta_3(e^{-\pi/t}).$$
• The Mellin transform of $e^{-\alpha t}$ for $\alpha \neq 0$ is $M_s(e^{-\alpha t}) = \Gamma(s)\alpha^{-s}$, so (letting $q = e^{-\pi t}$) we have

\[
L_3(s) = 3 \sum_{n,m,p \in \mathbb{Z}} \frac{n^2}{(n^2 + m^2 + p^2)^{s+1}}
\]

\[
= \frac{3\pi^{s+1}}{\Gamma(s+1)} \sum_{n,m,p \in \mathbb{Z}} n^2 M_{s+1}(q^{n^2+m^2+p^2})
\]

\[
= \frac{3\pi^{s+1}}{\Gamma(s+1)} M_{s+1} \left( \sum_{n \in \mathbb{Z}} n^2 q^{n^2} \theta_3^2(q) \right)
\]

\[
= \frac{3\pi^{s+1}}{\Gamma(s+1)} \sum_{n \in \mathbb{Z}} n^2 \int_0^\infty e^{-n^2\pi t} \theta_3^2(e^{-\pi/t}) t^{s-1} dt.
\]

So $L_3(s) =$

\[
\frac{3\pi^{s+1}}{\Gamma(s+1)} \sum_{n,m \in \mathbb{Z}} n^2 r_2(m) \int_0^\infty e^{-n^2\pi t - \frac{\pi m}{t}} t^{s-1} dt
\]

\[
+ \frac{3\pi^{s+1}}{\Gamma(s+1)} \sum_{n \in \mathbb{Z}} n^2 \int_0^\infty e^{-n^2\pi t} t^{s-1} dt.
\]

(14)
The first term of (14) is
\[
= \frac{6\pi^{s+1}}{\Gamma(s + 1)} \sum_{n=1}^{\infty} n^2 \sum_{m=1}^{\infty} r_2(m) \int_{0}^{\infty} e^{-n^2\pi t - \frac{\pi m}{t} t^{s-1}} dt
\]
\[
= \frac{12\pi^{s+1}}{\Gamma(s + 1)} \sum_{m=1}^{\infty} r_2(m) m^{s/2} \sum_{n=1}^{\infty} \frac{1}{n^{s-2}} K_s(2\pi n \sqrt{m})
\]
by (13) and the second term is
\[
= \frac{6\pi^{s+1}}{\Gamma(s + 1)} \sum_{n=1}^{\infty} \frac{1}{n^{2s-2} \pi^s} \int_{0}^{\infty} e^{-x x^{s-1}} ds
\]
\[
= \frac{6\pi}{s} \zeta(2s - 2).
\]
\[\text{This proves the following result:}\]
\[
L_3(s) = \frac{6\pi}{s} \zeta(2s - 2) + \frac{12\pi^{s+1}}{\Gamma(s + 1)} \sum_{m=1}^{\infty} r_2(m) m^{s/2} \sum_{n=1}^{\infty} \frac{1}{n^{s-2}} K_s(2\pi n \sqrt{m}).
\]
\[\text{This corresponds to Madelung’s constant.}\]
The second term of (15) can be rewritten as
\[
\frac{12\pi^{s+1}}{\Gamma(s + 1)} \sum_{k > 0} k^{s/2} K_s(2\pi \sqrt{k}) \sum_{n^2 | k} \frac{r_2(k/n^2)}{n^{2s-2}}.
\]

Moreover, these Bessel functions are elementary when \( s \) is a half-integer. Most nicely, for ‘jellium’, which is the Wigner sum analogue of Madelung’s constant, we have
\[
L_3(1/2) = -\pi + 3\pi \sum_{m > 0} \frac{r_2(m)}{\sinh^2(\pi \sqrt{m})},
\]
and the exponential convergence is entirely apparent.
But none of this seems to help with $R_3(s)$!

We have a corresponding formula for $L_N(s)$, for all $N \geq 2$, in which we obtain a Bessel-series in $r_{N-1}(m)$:

\[
L_N(s) = \sum_{n>0} \frac{r_N(n)}{n^s} = 2N \frac{\Gamma(s - \frac{N-3}{2}) \pi^{\frac{N-1}{2}}}{\Gamma(s + 1)} \zeta(2s - N + 1) + \frac{4N \pi^{s+1}}{\Gamma(s + 1)} \sum_{m>0} \frac{m^{\frac{1}{2}s}}{m^{\frac{N-3}{4}}} r_{N-1}(m) \times \sum_{n>0} \frac{n^{\frac{N+1}{2}}}{n^s} K_{s - \frac{N-3}{2}}(2n\pi\sqrt{m}).
\]
There is a puissant formula for $\theta_2^3$ due to Andrews (1986). It is

$$\theta_2^3(q) = 8 \sum_{n=0}^{\infty} \sum_{j=0}^{2n} \left( \frac{1 + q^{4n+2}}{1 - q^{4n+2}} \right) q^{(2n+1)^2-(j+1/2)^2}.$$ 

It shows almost immediately, Gauss’s result that every odd number is a sum of three triangular numbers.

Lamentably, we have not been able to use it to study $R_3$, or even $L_3$ any further than was recently achieved by Crandall.
Twelve and Twenty-four Squares

- Explicit ‘divisor’ formulae are well known:

\[ r_{12}(n) = 8(-1)^{n-1}\sum_{d \mid n}(-1)^{d+n/d}d^5 + 16\omega(n) \]

\[ r_{24}(n) = \frac{16}{691}\sigma_{11}^*(n) \]

\[ + \frac{128}{691}\left((-1)^{n-1}1259\tau(n) - 512\tau\left(\frac{1}{2}n\right)\right) \]

Here

\[ \sigma_{11}^*(n) = \sum_{d \mid n}d^{11} \]

if \( n \) is odd and

\[ \sigma_{11}^*(n) = \sum_{d \mid n}(-1)^d d^{11} \]

if \( n \) is even, and

\[ q((1 - q^2)(1 - q^4)(1 - q^6)\cdots)^{12} = \sum_{n=1}^{\infty} \omega(n)q^n \]

and

\[ q((1 - q)(1 - q^2)(1 - q^3)\cdots)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n. \]
• We record these representations because, while \( N = 12 \) and \( N = 24 \) (due to Ramanujan) are the next most accessible even cases, neither directly leads to an appropriate closed form for \( L_N \) let alone for \( R_N \).

◊ This is thanks to the impediment offered by \( \omega \) and \( \tau \) respectively: which encode knowledge, via the \textit{Jacobi triple-product}, of all the representations of \( n \) as a sum of 4 or 8 squares.

• The divisor functions do produce appropriate L-function representations. Thus, using Ramanujan’s \( \zeta \)-function

\[
\rho_{24}(s) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \left( 1 - \tau(p)p^{-s} + p^{11-2s} \right)^{-1},
\]

it transpires that \( \tau \) is multiplicative, with the preceding lovely Euler product.
• Additionally,

\[
L_{24}(s) = \sum_{n=1}^{\infty} \frac{r_{24}(n)}{n^s}
\]

\[
= \frac{16}{691}(2^{12-2s} - 2^{1-s} + 1)\zeta(s)\zeta(s-11)
\]

\[
+ \frac{128}{691}(745 \cdot 2^{4-s} + 259(1 + 2^{12-2s}))\rho_{24}(s).
\]

Similarly, with \( \rho_{12}(s) := \sum_{n=1}^{\infty} \frac{\omega(n)}{n^s} \) one has

\[
L_{12}(s) = \sum_{n=1}^{\infty} \frac{r_{12}(n)}{n^s}
\]

\[
= 8(1 - 2^{6-2s})\zeta(s)\zeta(s-5) + 16\rho_{12}(s).
\]

◊ Finally, we note that the Rankin provided an ‘almost closed form’ for

\[
\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^s} = \prod_p \left( 1 + \tau^2(p)p^{-s} - p^{22-2s} - \frac{2\tau^2(p)p^{-s}}{1 + p^{11-s}} \right)^{-1}.
\]
REFERENCES


• These and other references are available at www.cecm.sfu.ca/preprints/

◊ Quotations at jborwein/quotations.html

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