Abstract. John Barrow wrote recently in *Pi in the Sky: counting, thinking and being*

“today it is not unexpected to find the `computer’ or the `program’ as the central paradigms in our attempt to interpret the Universe,”

and he observed that

“the concept of experimental mathematics has begun to take on a new and more adventurous complexion.”
A dramatic “re-experimentalization” of Mathematics is indeed taking place. Driven by advances in hardware, software and theory, the computer takes on a laboratory role for pure and applied mathematicians; a role which in the eighteenth and nineteenth centuries the physical sciences played much more fully than in our century. In this process the role of proof in Mathematics (as known from Euclid on) is under siege and the subject may well be entering a much more inductive pseudo-empirical phase. It is this Kuhnian paradigm-shift, both attractive and threatening, about which I wish to talk.

Reviewing Barrow in *Science* we wrote:

“A pervasive use of the computer to attempt to interpret mathematics rather than just the Universe is surprisingly new. Mathematicians invented computers and then largely ignored them for several decades. It is only recently, with the advent of
really comprehensive symbolic manipulation or computer algebra packages, that computers have come of mathematical age."

It is only slightly an exaggeration to say that the computer can now obtain a good undergraduate honours degree in Mathematics.

New subjects such as fractal geometry, turbulence and chaotic dynamical systems have sprung up. Classical subjects like number theory, geometry and logic have received new infusions. Boundaries between mathematical physics, mathematical biology, and pure mathematics are more blurred than in many generations. Several recent Fields medals - our Nobel prizes - have gone for work which is perhaps pure mathematics, perhaps applied mathematics, perhaps physics.
I aim to illustrate the radical impact that the computer - with the internet - is having on mathematics and the way mathematicians do mathematics now and in the near future.

• This is an impossible task for one lecture...so
• What do I intend to do?

1. Illustrate Mathematics on the Internet while discussing:

2. The main mathematical ways of thinking
   - Geometry ( & Topology)
   - Algebra
   - Analysis (Zeeman’s anecdote)

3. Changing paradigms

4. Philosophies of Mathematics

5. Common misperceptions about mathematics

6. Proof versus truth and reliability of knowledge

7. Ideas of experiment in Mathematics
Back to the beginnings

500 BC. The Greeks were Geometers “measurers of the land”....

Harmony and Proportion:
The Golden Mean: 0.61803....

\[
\frac{1+x}{1} = \frac{1}{x}
\]

1000 AD. The Arabs developed Algebra (al--jabr) “bone-setting” or unifying...

• pictures came before equations

\[
x^2 + x = 1 \quad \Rightarrow \quad x = \frac{\sqrt{5}-1}{2}
\]

* In MAPLE this becomes···
• Returning to the Geometric way...

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots + = 1
\]

* In MAPLE this becomes…

• Algebra and then Analysis
The 3 problems of antiquity…

• The Greeks liked to “construct” using **protractor** (compass) and **straight-edge** (ruler) alone.

Bisecting an angle:

![Diagram of bisecting an angle]

we were all taught to do...
1. **Double the volume of a cube.**

(The Delian oracle problem: to stop a plague in 429 BC)

Volume = $x^3$  \hspace{1cm} Volume = 1

In math terms we must:

find $x$ by our ruler and compass rules:

$x^3 = 2 \quad (x = \sqrt[3]{2})$
2. Square the circle: “Quadrature”

In math terms we must:

find $x$ by our ruler and compass rules:

$$x^2 = \pi \quad (x = \sqrt{\pi})$$

- Aristophanes’ “The Birds” (415 BC) scorns τετραγωσιζειν - the circle squarers: people trying the impossible

- But much of mathematics has been dedicated profitably to the impossible.
3. Trisect an angle.

In math terms we must:

find $\alpha$, $\cos(\alpha)$, by our ruler and compass rules (we know $3\alpha$):

$$\cos(3\alpha) = 4\cos^3(\alpha) - 3\cos(\alpha)$$

• for $20^\circ$ this means solving

$$8x^3 - 6x - 1 = 0$$
• In fact all three problems are **impossible** by Greek rules.

• This was only proved in the 19th Century (2,500 years later!)

• The French Academy stopped accepting trisections centuries earlier!

• The problems are easy if you allow “better machines”.
• **Shifting Paradigms**

Newton → Einstein

Watchmaker → Blind Watchmaker

Clockwork → Universe as hardware, Physical laws as software

• Mathematicians and theoretical computer scientists now ask questions that computers make natural.

• They demand/accept different answers. (A century ago a “computer” was a person.)
• What are some of the principal Philosophies of Mathematics?

Crudely:

A. Platonism

• A new (1935) name for an old idea: mathematics is discovered, numbers exist.

• The view of most working researchers in the intellectual salt mines.

• Gödel’s results shook the foundations but hardly moved the building

B&C. Logicism and Formalism (1870-1935: Frege, Russell and Whitehead, Hilbert)
• Spurred by the famous 19C paradoxes of language and naive set theory: attempts to build up everything “from first principles” or purely formally (in response to intuitionism).

• does the book of all books not listing themselves in its index list itself?

D+E. Constructivism & Intuitionism (Kronecker, Poincaré, Brouwer, Lakatos).

• Certain Aristotelian logical principles are challenged: the law of the excluded middle: “A or not A”. It is demanded that proofs be broken down (in principle) into intuitively comprehensible components.

• Constructivism is the Missouri (”show me”) version of mathematics

• Computers change the battle (“and or gates” like to know which occurs).
Two examples:

**Theorem.** There exist two irrational numbers $\alpha$ and $\beta$ such that $\alpha^\beta$ is rational.

**Platonic Proof.** Consider

\[ (\sqrt{2} \cdot \sqrt{2})^2 = (\sqrt{2})^2 = 2 \]

Now 2 is rational.

a) If $\sqrt{2} \cdot \sqrt{2}$ is rational let $\alpha = \beta = \sqrt{2}$.

b) If $\sqrt{2} \cdot \sqrt{2}$ is irrational let $\alpha = \sqrt{2}$ and $\beta = \sqrt{2}$.

By the principle of the excluded middle we are done. QED

- In fact case b) holds but the proof is very deep (Gelfond-Schneider, 1935) and non constructive!
2. Consider the infinite sequence \( \{x_n\} \) which is zero in every place except the first time 012456789 begins in the decimal expansion of \( \pi \). (Brouwer-Heyting)

**Q:** Does this sequence converge?

**A:** ‘Yes’ for Platonists: \( \{x_n\} \) is zero for large \( n \).

‘No’ for Constructivists and for Intuitionists: how big must \( n \) be before \( |x_n| < \frac{1}{2} \) ? (\( n > \text{five billion!} \)).

• Eventually (soon?) the sequence will occur in a super computation of \( \pi \). Will the sequence then begin to converge or always have converged?

* More sophisticated less avoidable versions of these examples occur frequently
• **Common (Mis)perceptions**

• “Mathematics is sullen protein:”

G.N. Watson of some of Ramanujan’s formulae describes:

“a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing “Day,” “Night,” “Evening,” and Dawn” which Michelangelo has set over the tomb of Guiliano de’ Medici and Lorenzo de’ Medici.”

• “Truth equals proof:”

Godel (1935): in any mathematical system sophisticated enough to encode (Peano) arithmetic there are true but unprovable statements.

• “Pure equals useless:”
My next lecture!

• “Obvious equals easy:”

Every simple closed curve has an inside and an outside (Jordan, 19C)

• “Mathematics is done like in math books (Definition/Lemma/Proof):”

or

• “Mathematics is cut and dried:”

… in what follows …
• **Proof versus Truth**

  • Heuristics and Probability
  
  • Visualisation
    (picture = $10^3$ words)
  
  • Truth is Beauty
    (Keats, Blake, Gödel)
  
  • Decidability and Consciousness
    (Penrose, “Emperor’s New Mind”)
  
  • The computer as Anthill
    (Lenstra, Leyland and RSA 129)
  
  • No solution: Projective plane $P_{10}$
or Fermat’s Last Theorem
    (Lam vs Wiles)
  
  • Classifying Finite Simple Groups
    (12 volumes: Feb 95 Notices of AMS)
• (Mathematical) Experiments:

  Peter Medawar’s four types:

  - Mathematicians Explore:

    i) In the original **Baconian sense**, an experiment is a contrived, as opposed to a natural, experience or happening--is the consequence of `trying things out` or even of merely messing about.

    - Mathematicians Demonstrate:

      ii) **Aristotelian Experiments** are demonstrations such as applying electrodes to a frog’s legs and lo, the leg kicks. Precede the presentation of dinner to a dog by a bell and soon the bell will make the dog dribble.

      These are tailored to demonstrate the theorems: rather than being used to devise and revise the theorems.
iii) **Kantian Experiments** generated `classical non-Euclidean geometries’ (hyperbolic, elliptic) by replacing Euclid’s axiom of parallels (or something equivalent to it) with alternative forms.’

- but Mathematicians have rarely “Experimented”:

iv) A **Galilean Experiment** is a critical experiment -- one that discriminates between possibilities.

("Advice to a Young Scientist")
• **Experiment, Right or Wrong.**

“Establishing the validity of experimental results” (Allan Franklin):

[1.] Experimental checks and calibration, in which the apparatus reproduces known phenomena.

[2.] Reproducing artifacts that are known in advance to be present.

[3.] Intervention, in which the experimenter manipulates the object under observation.

[4.] Independent confirmation using different experiments.

[5.] Elimination of plausible sources of error & alternative explanations of the result.

[6.] Using the results themselves to argue for their validity.
[7.] Using an independently well-corroborated theory of the phenomena to explain the results.

[8.] Using an apparatus based on a well-corroborated theory.

[9.] Using statistical arguments.

• We have been trying to fit these ideas into a large-scale computational & statistical analysis of the distribution of digits and continued fractions of numbers: a true Galilean Experiment.

(CECM reprint available)
• Examples

# Prime Number Theorem (1896):

The number of primes less than \( x \) “behaves like”

\[
\pi(x) \approx \frac{x}{\log(x)} \approx \int_2^x \frac{dt}{\log t} =: \text{Li}(x)
\]

conjectured after much computation - by Gauss, Meissel (1870) and others.

\[
\begin{align*}
\pi(10^6) &= 78,498 \\
\text{Li}(10^6) &= 78,628 \\
\text{Li}(10^9) &= 50,849,235 \\
\pi(10^9) &= 50,847,478
\end{align*}
\]

• Viewed as the acme of 19th C Math.

• The *Riemann hypothesis* is the refined version of ‘≈’.

• Computing \( \pi_2(x) \) found Pentium bug.
# Four colour theorem (Guthrie 1850)

"every map can be coloured with four colours"

("four colors suffice," Appel and Haken 1976).

- The computation was only finally independently confirmed in the last few years.
A projective plane of order 2 (7 points, & lines, each on 3 of the other):

From the “Internet Math FAQ”

Q: Does there exist a projective plane of order 10? \[10^2+10+1=111\]

More precisely: Is it possible to define 111 sets (lines) of 11 points each so that:
For any pair of points there is precisely one line containing them both and for any pair of lines there is only one point common to them both?
A: Analogous questions with \( n^2 + n + 1 \) and \( n + 1 \) instead of 111 and 11 have been positively answered only in case \( n \) is a prime power.

For \( n=6 \) it is not possible, more generally if \( n \) is congruent to 1 or 2 mod 4 and can not be written as a sum of two squares, then an FPP of order \( n \) does not exist.

The \( n=10 \) case has been settled as not possible either by Clement Lam (Concordia). As the “proof” took several years of computer search (the equivalent of 2,000 hours on a Cray-1) it can be called the most time-intensive computer assisted single proof. The final steps were ready in January 1989.
References.


• A story with high Canadian content!

• Why believe the computer when it said “NO”? Indeed with vanishingly small probability of error we should.
# 1650: Fermat’s last theorem for n>2

\[ a^n + b^n = c^n \]

has no positive integer solution (Wiles, 1994?) but by computer

\[ 27^5 + 84^5 + 110^5 + 133^5 = 144^5 \]

(1966)

\[ 2682440^4 + 15365639^4 + 18796760^4 \]
\[ = 20615673^4 \]

(1987, Elkies+elliptic curves) and

\[ 95800^4 + 217519^4 + 414560^4 = 422481^4 \]
# Triple product & Jacobi identities

* Let us look in MAPLE at the important infinite product

\[ \Pi := (1-q)(1-q^2)(1-q^3)\cdots(1-q^n)\cdots \]

and see how symbolic computation aids discovery of celebrated formulae for the series for \( \Pi \) and \( \Pi^3 \).
# Probabilistic methods

* primality testing and public key codes

* zero equivalence: are two expressions equal?

• a recent paper on Ramanujan’s work contains

“it remains to show

\[
\sqrt[3]{11+6\sqrt{32}} + \sqrt[3]{9+6\sqrt{32}} \quad =
\]

\[
12 (\sqrt[4]{4+\sqrt{3}} + \sqrt[4]{3})
\]

• high precision numerical check?

• finite fields probabilistic check?

• exact symbolic verification?
# Mathematics by analogy

- Evolution
- J. J Sylvester’s molecules (1870)
- Catastrophe Theory
- Deterministic Chaos
- Simulated Annealing
- Genetic Algorithms
- DNA Computing
- Quantum Computers
Two final provocative views:


“It is very time consuming to solve a system of linear equations with symbolic coefficients. By plugging in specific values for $n$ and other parameters if present, one gets a system with numerical coefficients, which is much faster to handle. Since it is unlikely that a random system of inhomogeneous linear equations with more equations than unknowns can be solved, the solvability of the system for a number of special values of $n$ and the other parameters is a very good indication that the identity is indeed true.

*It is a waste of money to get absolute certainty, unless the conjectured identity in question is known to imply the Riemann Hypothesis.*”
Goldbach’s conjecture (1742): every even number is the sum of two primes.

Abstract of the future. We show in a certain precise sense that the Goldbach conjecture is true with probability larger than 0.99999 and that its complete truth could be determined with a budget of 10 billion.

“Using software written in Mathematica that runs on an IBM RS/6000 workstation, I constructed a perverse 200-page algebraic equation with a parameter $N$ and 17,000 unknowns:

$$\text{LeftHandSide}(N) = \text{RightHandSide}(N).$$

For each whole-number value of the parameter $N$, ask whether this equation has a finite or an infinite number of whole number solutions. The answers escape the power of mathematical reason because they are completely random and accidental. This work is an extension of famous results of Gödel and Turing using ideas from a new field called algorithmic information theory.
I believe that elementary number theory and the rest of mathematics should be pursued more in the spirit of experimental science, and that you should be willing to adopt new principles. I believe that Euclid’s statement that an axiom is a self-evident truth is a big mistake.

The Schrödinger equation certainly isn’t a self-evident truth! And the Riemann hypothesis isn’t self-evident either, but it’s very useful. A physicist would say that there is ample experimental evidence for the Riemann hypothesis and would go ahead and take it as a working assumption.

*What is the Riemann hypothesis?* There are many unsolved questions involving the distribution of the prime numbers that can be settled if you assume the Riemann hypothesis. Using computers people check these conjectures and they work beautifully. They’re neat formulas but nobody can prove them.
A lot of them follow from the Riemann hypothesis. To a physicist this would be enough: It’s useful, it explains a lot of data. Of course a physicist then has to be prepared to say “Oh oh, I goofed!” because an experiment can subsequently contradict a theory. This happens very often. In particle physics you throw up theories all the time and most of them quickly die. But mathematicians don’t like to have to backpedal. But if you play it safe, the problem is that you may be losing out, and I believe you are.”
• My own view:

“Mathematics is ultimately about highly reliable mathematical knowledge. Proof and experiment, introspection and computation, probability and certainty will co-exist in varying balance.

But, without serious discussion of the changes underway, we will lose the benefits of our tradition without reaping the rewards of the new regime.

“It is true because Von Neumann, Maple or Mathematica says so...”

is the likely and unsatisfactory consequence of inattention.”