Part II: Regularity, Global Normality and Tangency

Overview

Part II

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1. Introduction to Part II

Linear (metric) regularity. Recall that $C_1$ and $C_2$ are linearly regular if \( \exists \kappa > 0 \) such that

\[
d(x, C) \leq \kappa \max\{d(x, C_1), d(x, C_2)\}, \quad \forall x \in X.
\]

Metric regularity – even for non-convex sets – provides the cleanest way to establish tangency and normality formulae.

Deutsch and his coauthors call this CHIP and strong CHIP and again analyze this in the context of best approximation.

2. Regularity, Normality and Relative Openness

- For \( 0 < \alpha < 1 \) and \( p = \alpha^{-1} \), define a set in \( \mathbb{R}^3 \) by
  \[
  S_p = \{(x, y, z) : |y| \leq \gamma_p x^{1-\alpha}, x \geq 0, z \geq 0\}
  \]
  where \( \gamma_p = \sqrt{p(p-1)^{\alpha/2}} \). Then \( S_p \) is self-dual: \( S_p^* = S_p \).

- \( K_2 \) is the cone \( \{(y, z, w) | |y| \leq \sqrt{2wz}, z \geq 0\} \). That is \( S_2 \) with coordinates permuted.

Normality without openness/regularity

Define closed cones \( A \) and \( B \) in \( X = \mathbb{R}^4 \) as

\[
A = (0 \times K_2) + (S_3 \times 0) \quad B = 0 \times \mathbb{R}^3.
\]

Then \( A \cap B = 0 \times K_2 \) and

\[
A^+ = (\mathbb{R} \times K_2) \cap (S_3 \times \mathbb{R}).
\]

Using the Krein–Rutman theorem, \( K = A^+ + B^+ \) is closed.

- Let \( P \) be the projection on \( Y = \mathbb{R}^3 \) defined by \( P(x, y, z, w) = (y, z, w) \). We show that \( P(K) = K_2 \) or equivalently
  \[
  A^+ + B^+ = \mathbb{R} \times K_2.
  \]

Indeed: \( K_2 \subset P(\mathbb{R} \times K_2) \subset P(A^+) \subset K_2 \). For the first inclusion, consider the cases \( z = 0 \) and \( z > 0 \) separately.

- In particular, \( P(K) \) is closed, as is \( K + M \).

Now \( A^+ + B^+ = (A \cap B)^+ (= \mathbb{R} \times K_2 \) in this case) always implies the global normal cone formula (GNF):

\[
N_A(x) + N_B(x) = N_{A \cap B}(x)
\]

for all \( x \in A \cap B \). By duality, the global tangency formula (GTF):

\[
T_A(x) \cap T_B(x) = T_{A \cap B}(x)
\]

holds for all \( x \in A \cap B \).
• Note, since $A$ is convex $N_A(x) = -(A - x)^+$ and $T_A(x) = \text{co} \text{lin}(A - x)$. We may verify $T_A(x) \cap T_B(x) = T_{A \cap B}(x)$ directly.

• While $P(K)$ is closed, there is no constant $\varepsilon > 0$, so that

$$\varepsilon B \cap P(K) \subset P(K \cap B_X).$$

This shows that the open mapping theorem may fail when $P(K)$ is closed but not linear.

• For cones $S$ and $K$

$$\text{GNF} \Leftrightarrow (S \cap K)^+ = S^+ + K^+$$

while

$$\text{BLR} \Leftrightarrow \varepsilon B \cap (S \cap K)^+ \subset B \cap S^+ + B \cap K^+$$

for some $\varepsilon > 0$.

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**variants on the (GNF) example.**

Consider the abstract linear program:

$$h(b) := \min \{ x : P(x, v) = b, (x, v) \in K \}$$

with $P$ and $K$ as above.

• Observe the value function has dom $h = K_2$ and that explicitly for $(y,z,w) \in S_2$ one has $h(y,z,w) = 0$ if $z = 0$ while for $z > 0$

$$h(y,z,w) = \frac{y^2}{z^3}$$

Thus $h$ is lower semi-continuous with a closed domain but $h|K_2$ is not continuous at zero.

• One can easily show that the infimum is indeed attained.

• Let $b \in K_2$ be fixed. Consider the dual program. There is no duality gap and the dual supremum is attained: as computing $\partial h(b)$ shows.

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**In fact:** $P(x, t^2, t^3, t) = (t^2, t^3, t) \Rightarrow x \geq \gamma_3^{-1} t$ if $(x, t^2, t^3, t) \in K$.

• Equivalently, with $M = B^+$, there is no constant $\varepsilon > 0$, so that

$$\varepsilon B \cap (K + M) \subset (K \cap B_X) + (M \cap B_X),$$

despite $K + M$ being closed, $K$ being a closed convex pointed cone with interior and $M$ being a single line.

• Consider $(x, v) = (\gamma_3^{-1} t^2, t^3, 0) \in A$. Observe that $d_B(x, v) = \gamma_3^{-1}$ and $d_A(x, v) = 0$, and that $d_{A \cap B}(x, v) \equiv d_{S_2}(v) \to \infty$ as $t \to \infty$.

• Thus, linear regularity fails for $A$ and $B$ while the global normality and tangency consequences (central to optimization) are valid.

• If and only if $X$ has dimension $\geq 4$.

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**3. Normality and Tangency**

**Remaining Goal:** Find convex sets (cones in one higher dimension) $A$ and $B$ in $\mathbb{R}^4$ for which (GTF) holds but not (GNF).

• Define $A :=$

$$S_2 \cap \{(x, y, z) : (x - 1)^2 + z^2 \leq 1, 0 \leq x \leq 1, 0 \leq z \}$$

and

$$B := 0 \times Q$$

where as before

$$S_2 := \{(x, y, z) : y^2 \leq 2xz, z \geq 0, x \geq 0 \}.$$

while $Q$ is a quarter circle

$$Q := \{(y, z) : z^2 + (y - 1)^2 \leq 1, 0 \leq y \leq 1, 0 \leq z \} = \{(y, z) \in \mathbb{R}^2 : 2y \geq y^2 + z^2, y \leq 1, 0 \leq z \}.$$
• The key to the geometry is:
  
  \( \diamond \) Suppose \( q = (y, z) \in \text{bd}(Q) \). Then exactly one of the following two alternatives holds.

  1. \( z = 0 \) and \( 0 \leq y \leq 1 \). In this case,
     \[
     T_Q(q) = \begin{cases} 
     \mathbb{R}_+ \times \mathbb{R}_+, & \text{if } y = 0; \\
     \mathbb{R} \times \mathbb{R}_+, & \text{if } 0 < y < 1; \\
     \mathbb{R}_- \times \mathbb{R}_+, & \text{if } y = 1.
     \end{cases}
     \]

  2. Either \( y = 1 \) and \( 0 < z \leq 1 \), or \( 0 < y < 1 \) and \( z = \sqrt{2y - y^2} \). In either case,
     \[
     Q \cap \mathbb{R}_+ : q = [0, 1] : q
     \]
     and
     \[
     q \notin T_Q(q), \text{ but } -q \in T_Q(q).
     \]

Now set
\[
C := \text{conv}(A \cup B), \quad Y := 0 \times \mathbb{R}^2.
\]

[4. A Conical Open Mapping Theorem]

• Somewhat surprisingly there is an open mapping theorem on cones:

**Theorem.** (Conic OMT) Let \( T : X \to Y \) be linear and bounded. Suppose \( K \subset X \) is a closed convex cone and that \( S = T(K) \) is closed and that \( T|K \) is injective. Suppose that \( K \) is boundedly relatively weakly compact as holds if either (a) \( X \) is reflexive or (b) \( K \) has a weakly compact base.

Then \( T \) is relatively open at the origin on \( K \).

• When \( T(K) \) is linear we can quotient out the injectivity and reclaim the classic result. Is the conic OMT valid in all Banach spaces?

• Properties of \( A + B \) are equivalent to those of \( T(K) \) — in higher dimensions.

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**Theorem.** The sets \( A, B, C \) are compact, convex, and nonempty in \( \mathbb{R}^3 \). Also,

1. \( T_A(0) = S_2 \).
2. \( A \cap B = A \cap Y = \{0\} \).
3. \( C \cap Y = B \).
4. \( T_C(b) \cap Y \subseteq T_B(b), \) for every \( b \in B \).
5. \( T_C(b) \cap T_B(b) = T_B(b), \) for every \( b \in B \).
6. \( (0, -1, 0) \in N_B(0) \setminus [N_C(0) + N_Y(0)]. \)

Thus, \( \text{GNF} \) fails for \( C \) and \( Y \) but \( \text{GTF} \) holds.

• Is there a conical example in \( \mathbb{R}^3 \)?

[5. Conclusion to Part II]

• Observe that in the (GNF) example,

\[
\varepsilon_1, \varepsilon_3, \varepsilon_4 \in A \cap K
\]

where \( \varepsilon_i \) is the \( i \)-th unit vector. Hence \( A \) and \( K \) are neither smooth nor strictly convex.

\( \diamond \) Indeed, there are no such counter-examples if \( A \) is smooth away from the origin while \( B \) is a subspace. More generally,

• Suppose \( C_1, \ldots, C_m \) are finitely many closed convex sets in \( X \) with

\[
C := \bigcap_i C_i \neq \emptyset
\]

and \( x \in C \). We say that the collection \( \{C_1, \ldots, C_m\} \) is **intersection-closed at** \( x \), if whenever \( (c_n) \) is a sequence in \( C \) converging to \( \bar{c} \) and \( (y_n) \) are sequences converging to \( y \) with

\[
y_i, n \in T_{C_i}(c_n), \quad \forall m, i \Rightarrow y_i \in T_{C_i}(\bar{c}), \forall i.
\]
If $D$ is a closed nonempty subset of $C$ and \{$C_1, \ldots, C_m$\} is intersection-closed at every point in $D$, then we say that \{$C_1, \ldots, C_m$\} is intersection-closed on $D$.

**Theorem. (Conditions for GTF $\Rightarrow$ BLR).** Suppose $C_1, \ldots, C_m$ are finitely many closed convex sets in $\mathbb{R}^n$ and are intersection-closed on $C$ with
\[ C := \bigcap_i C_i \neq \emptyset. \]
In particular, this holds when (i) $C$ is singleton; or (ii) each $C_i$ is smooth on $C$.

Then \{$C_1, \ldots, C_m$\} is boundedly linearly regular if and only if the tangency (resp. normal) formula holds globally.

**REFERENCES**


2. H.H. Bauschke, J.M. Borwein and Wu Li, “On the strong conical hull intersection property, bounded linear regularity, Jameson’s property (G), and applications to convex optimization,” [CECM Preprint 97-101].


- Recall, BLR $\Rightarrow$ GNF $\Rightarrow$ GTF and ‘interiority’ or polyhedrality $\Rightarrow$ BLR.

- This is a finite dimensional result: in infinite dimensions you can find closed subspaces $M$ and $N$ with $M^\perp + N^\perp$ not closed. Hence,
\[ M^\perp + N^\perp \neq (M \cap N)^\perp. \]